Filters

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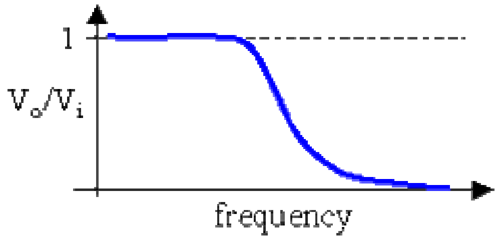
Introduction

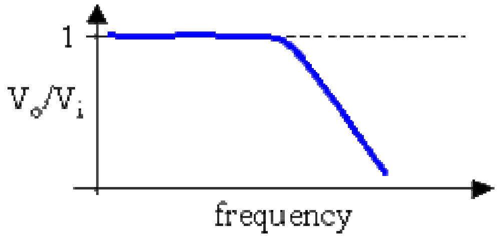
- A filter is used to modify a voltage signal, usually to get rid of some unwanted frequencies in the signal. In general, consider a filter operating on some input voltage:

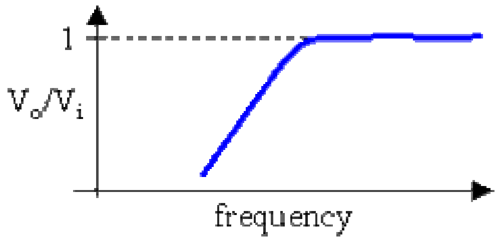
  Here, V_i is the input voltage, and V_o is the output voltage.
- There are four basic types of filters:
  - A ***low-pass filter*** lets low frequencies go through or *pass*, but attenuates high frequencies. V_o/V_i is plotted as a function of frequency below:

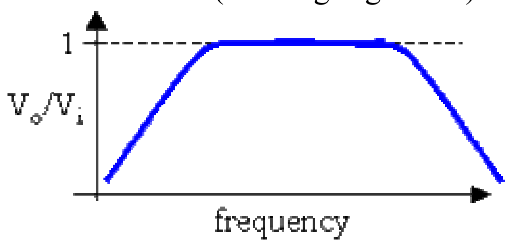
    Usually, logarithmic scales are used on both the horizontal and vertical axes in plots like this, i.e.

  - A ***high-pass filter*** lets high frequencies go through or *pass*, but attenuates low frequencies, as sketched below (with log-log scales):

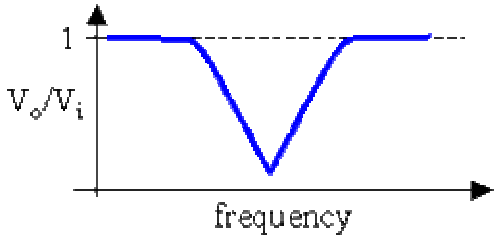
    

  - A ***band-pass filter*** is a combination of the above two. It lets a band of frequencies go through or *pass*, but attenuates both low frequencies and high frequencies, as sketched below (with log-log scales):

  - Finally, a ***band-stop filter*** is the exact opposite of a band-pass filter. It lets *all* frequencies go through or pass, except for some band of frequencies, which it

suppresses or *stops*, as sketched below (with log-log scales):

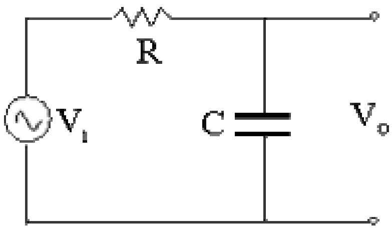


Electronic Filter Circuits

- In this section, some simple filter circuits are constructed and analyzed.
- Resistors, capacitors, and inductors are the only components used to construct these filters. The textbook describes filters created with operational amplifiers (op amps).

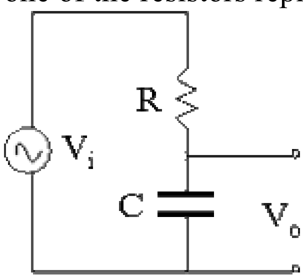
First-order low-pass filter

- A resistor and capacitor can be used together to create a simple low-pass filter circuit as shown:



- Consider an input voltage that is a simple sine wave of known frequency, f (or angular frequency $\omega = 2\pi f$), of the form
  $V_i = V_p \sin(2\pi ft) = V_p \sin(\omega t)$

  where V_p is the peak voltage or amplitude of the signal, which is also equal to $|V_i|$. Note that the peak-to-peak amplitude is twice V_p.
- Note also that for simplicity, there is assumed to be no DC offset in the input signal.
- Output voltage V_o is measured by some device (voltmeter, oscilloscope, etc.). Note that it is always assumed that ***the device which measures output voltage V_o has infinite impedance***. I.e., the measuring device does not affect the circuit in any way. For example, it does not draw any current or cause any voltage drop. In other words, it is a ***non-intrusive*** measuring device.
- Modern digital multimeters, oscilloscopes, and PC data acquisition cards have huge, but not infinite impedance, so the above assumption is very good.
- To analyze the low-pass filter circuit, we think of it as a simple voltage divider, except with one of the resistors replaced by a capacitor.



- Instead of resistance, impedance is used to analyze this divider circuit.
- Recall that for the resistor, $\mathbf{Z}_R$ = R, and for the capacitor, $\mathbf{Z}_C = 1/i\omega C$, again using **bold** fonts for complex variables.

- Impedances add in series just like resistors, so

$$\mathbf{Z}_{total} = \mathbf{Z}_R + \mathbf{Z}_C = R + \frac{1}{i\omega C}$$

- Just as for a simple resistor voltage divider circuit, the output voltage here is equal to the input voltage times a linear fraction of the impedances as follows (note that the bold voltage is complex):

$$\mathbf{V}_o = V_i \frac{\mathbf{Z}_C}{\mathbf{Z}_R + \mathbf{Z}_C}$$

$$\mathbf{V}_o = V_i \frac{\frac{1}{i\omega C}}{R + \frac{1}{i\omega C}}$$

- After rearranging and multiplying and dividing by the complex conjugate of the denominator, the complex voltage can be split into its real and imaginary components

$$\mathbf{V}_o = V_i \frac{1}{1 + i\omega RC} = V_i \frac{1 - i\omega RC}{(1 + i\omega RC)(1 - i\omega RC)}$$

$$\mathbf{V}_o = V_i \left[\frac{1}{1 + (\omega RC)^2} - i \frac{\omega RC}{1 + (\omega RC)^2} \right]$$

- It turns out that the magnitude of $\mathbf{V}_o$ is the magnitude of the output voltage V_o itself, and the angle of $\mathbf{V}_o$ in the complex plane is the phase shift ϕ of V_o from the input signal V_i. Recall that the magnitude of a complex number is a real number called the ***modulus***. Mathematically, one finds the magnitude or modulus of a complex number by taking the square root of the sum of the real part squared and the imaginary part squared, i.e.

$$|V_o| = |\mathbf{V}_o| = \sqrt{\frac{1}{\left(1 + (\omega RC)^2\right)^2} + \frac{(\omega RC)^2}{\left(1 + (\omega RC)^2\right)^2}} \cdot V_i$$

$$= \sqrt{\frac{1 + (\omega RC)^2}{\left(1 + (\omega RC)^2\right)^2}} \cdot V_i$$

$$= \sqrt{\frac{1}{1 + (\omega RC)^2}} \cdot V_i$$

or, finally,

$$|V_o| = \frac{V_i}{\sqrt{1 + (\omega RC)^2}}$$

- The ***break frequency*** (also called the ***corner frequency*** or ***cutoff frequency***) for this RC circuit is defined as

$$\omega_b \equiv \frac{1}{RC}$$

Note that this is the *radian* frequency (in radians per second), not the physical frequency (in Hz).

- The ***physical break frequency***, f_b, can be defined as

$$f_b = \frac{\omega_b}{2\pi}$$

- Then, the magnitude of the output voltage becomes

$$|V_o| = \frac{V_i}{\sqrt{1 + \left(\frac{\omega}{\omega_b}\right)^2}}$$

- The ***phase shift*** of the output signal is found as follows:

$$\phi = \arctan\left(\frac{\text{imaginary component}}{\text{real component}}\right) = \arctan\left(\frac{\dfrac{-\omega RC}{1+(\omega RC)^2}}{\dfrac{1}{1+(\omega RC)^2}}\right)$$

or $\phi = \arctan(-\omega RC) = \arctan\left(-\frac{\omega}{\omega_b}\right)$

or finally, $\phi = -\arctan\left(\frac{\omega}{\omega_b}\right)$

- Note the following two equations:

$V_i = V_p \sin(\omega t) = |V_i|\sin(\omega t)$

$V_o = |V_o|\sin(\omega t + \phi)$

where $|V_i|$ is the peak amplitude of the input signal $(|V_i| = V_p)$, and $|V_o|$ is the peak amplitude of the output signal.

- Summarizing the equations for the output voltage signal for this first-order low-pass filter,

$$|V_o| = \frac{|V_i|}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^2}}$$

or $G \equiv \frac{|V_o|}{|V_i|} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{f}{f_b}\right)^2}}$

where $\omega_b = \frac{1}{RC}$ and $f_b = \frac{\omega_b}{2\pi}$

- Note that the ***gain***, G, of the filter is defined as the ratio of the peak amplitude of V_o to the peak amplitude of V_i. G can be defined in terms of either frequency, f, or angular frequency, ω. Also note that for this simple filter, G lies between 0 and 1.
- The phase shift of the output signal is

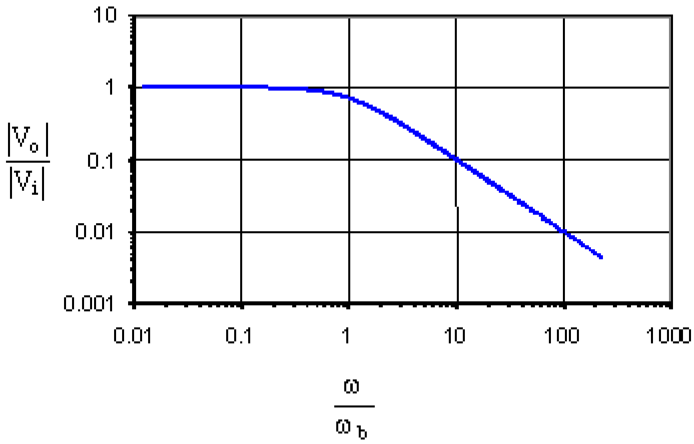
$\phi = -\arctan\left(\frac{\omega}{\omega_b}\right)$.

- What does all this mean physically, and why is this considered a low-pass filter circuit? Well, let us analyze what happens to both DC and AC voltage signals:
  - For a *DC signal*, $\omega = 0 \left(\frac{\omega}{\omega_b} = 0\right)$, and thus $|V_o| = |V_i|$ and $\phi = 0$. Thus $V_o = V_i$.
    I.e. ***the circuit does not affect a DC signal at all***.
  - For a *low frequency AC signal*, $\omega = \text{small} \left(\frac{\omega}{\omega_b} << 1\right)$, and thus $|V_o| \approx |V_i|$ and $\phi \approx 0$.
    Thus $V_o \approx V_i$.
    I.e. ***low frequencies pass through the circuit without much effect*** . (This is why it is called a low-pass filter, by the way.)
  - For a *high frequency AC signal*, $\omega = \text{large} \left(\frac{\omega}{\omega_b} >> 1\right)$, and thus $|V_o| \to 0$ and $\phi \to -\frac{\pi}{2} = -90^o$. Thus $V_o \to 0$ with a -90° phase shift.
    I.e. ***high frequencies are attenuated or filtered by the circuit***, and have a ***-90° phase shift***.

- For an *AC signal with a frequency exactly at the cutoff frequency*, $\omega = \omega_b \left(\frac{\omega}{\omega_b} = 1 \right)$, and thus $|V_o| = \frac{|V_i|}{\sqrt{2}} \approx 0.707|V_i|$ and $\phi = \arctan(-1) = -\frac{\pi}{4} = -45^o$. Thus V_o is attenuated by some 30% with a -45^o phase shift.

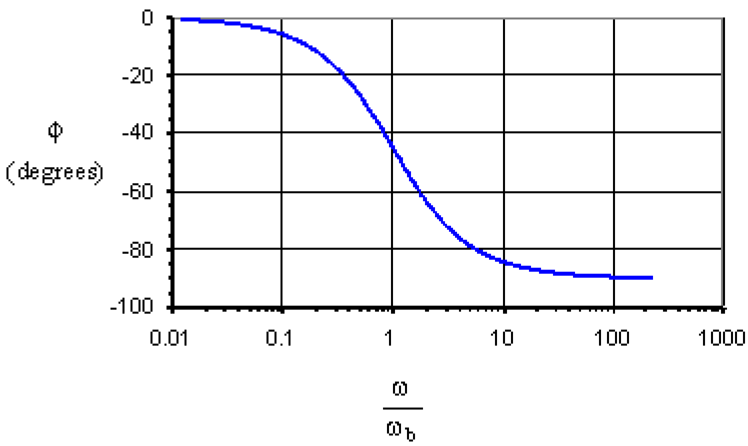
  I.e. ***the cutoff frequency is attenuated by a factor of*** $1/\sqrt{2}$.

- Below are summary plots for this first-order low-pass filter:
- First, the filter function itself, i.e. the ratio of the magnitude of V_o to that of V_i:



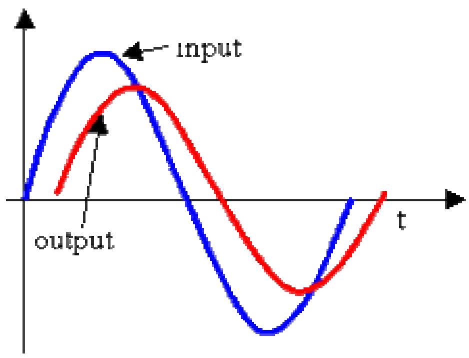
Notice how the low frequencies pass relatively unaffected, but the high frequencies get attenuated. Also notice that the drop-off with frequency looks linear on this log-log plot.

- Second, the phase angle:



The phase angle is zero (no phase shift) for very low frequencies, but falls off rapidly. Even at the corner frequency, the phase shift is significant, i.e. -45^o. For high frequencies the phase shift asymptotes to -90^o.

- This phase shift is illustrated in another way below, where the input and output signals are drawn for comparison:



Notice two things:

- The output is smaller in magnitude than the input
- The phase of the signal has shifted. Since the phase shift is *negative*, one can say that ***the output lags the input*** (i.e. the peak occurs at a later time).

Example problem

Given:

- Noise at 1000 Hz is superimposed on a "carrier" frequency of 10 Hz.
- It is desired to remove the noise so that only the carrier signal remains.

To do:

- Choose the break frequency (corner frequency or cutoff frequency) of the low-pass filter.
- **Solution**: Obviously, the break frequency must lie somewhere between 10 and 1000 Hz. If one picks a corner frequency too close to 10 Hz, some of the desired signal is attenuated. On the other hand, if the corner frequency is too big, the attenuation of 1000 Hz noise may not be enough. Let's pick 50 Hz as a reasonable first guess for the cutoff frequency.

- If a capacitor with capacitance of 0.1 microfarads is available, what resistor should be used?
- **Solution**:

$$R = \frac{1}{\omega_b C} = \frac{1}{2\pi f_b C}$$

$$R = \frac{1}{2\pi\left(50\frac{1}{s}\right)(0.1\text{x}10^{-6}\,F)}\left(\frac{F\cdot V}{C}\right)\left(\frac{C}{s\cdot A}\right)\left(\frac{\Omega\cdot A}{V}\right)$$

$$R = 3.18\text{x}10^{4}\ \Omega \approx\ 32\ k\Omega$$

Notice the unit conversions on the right side of the above equation.

- By what factor is the noise reduced with this filter?
- **Solution**: To answer this question, one must plug in the noise frequency (1000 Hz) and the cutoff or break frequency (50 Hz) into the equation for the gain, G of the filter. Here, frequency ratios are used rather than angular frequency ratios.

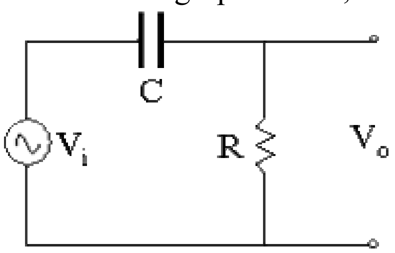
$$G \equiv \frac{|V_o|}{|V_i|} = \frac{1}{\sqrt{1+\left(\frac{f}{f_b}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{1000}{50}\right)^2}} = 0.04994 \approx 0.05$$

In other words, the noise is reduced by a factor of about 20.

- It is standard to express the gain of both amplifiers and filters in terms of ***decibels***, i.e.
  $G_{dB} \equiv 20\log_{10} G = 20\log_{10}\left(\frac{|V_o|}{|V_i|}\right)$
  For this example, GdB = 20log10(0.04994) = -26.0 dB.
- It is important to also calculate the gain imposed by this filter on the 10 Hz signal. (Recall that the 10 Hz signal is the *desired* signal here.) Using the same formulas as above, but with f = 10 Hz, one can calculate the gain of this low-pass filter at a frequency of 10 Hz, i.e. G = 0.9806, or GdB = 20 log10(0.9806) = -0.170 dB.
- Is this filter adequate? The answer depends on the application. With this filter, the noise at 1000 Hz is reduced by a factor of 20, while the desired signal itself (10 Hz) is reduced by about 2 percent. For most applications, this is fine. However, if more attenuation of the noise is required, or if a two percent reduction of the signal is too much, then this filter would not be adequate. In such a case, the engineer would choose a higher-order filter, as dicussed later.

First-order high-pass filter

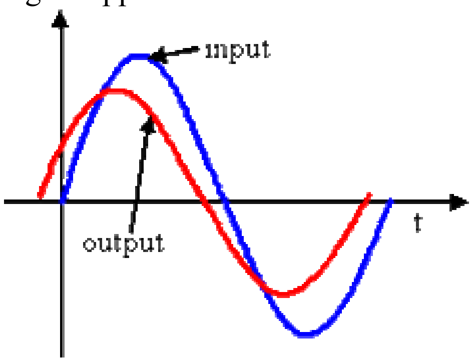
- A resistor and capacitor can be used together to create a simple high-pass filter circuit as well. For a high-pass filter, the resistor and capacitor switch positions as shown:



- The algebra to determine the filter function and the phase shift is similar to that performed above for the low-pass filter, and is not shown in detail here.
- Below is a summary of the equations for the output voltage signal for this first-order high-pass filter, noting that $V_i = V_p \sin(\omega t)$:
  $V_o = |V_o|\sin(\omega t + \phi)$
  $$|V_o| = \frac{|V_i|\cdot\left(\frac{\omega}{\omega_b}\right)}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^2}} = |V_i|\frac{1}{\sqrt{1+\left(\frac{\omega_b}{\omega}\right)^2}}$$
  $$G = \frac{|V_o|}{|V_i|} = \frac{1}{\sqrt{1+\left(\frac{\omega_b}{\omega}\right)^2}} = \frac{1}{\sqrt{1+\left(\frac{f_b}{f}\right)^2}}$$
  $$\phi = \arctan\left(\frac{\omega_b}{\omega}\right)$$
  $$\omega_b = \frac{1}{RC}$$
- Let us again analyze what happens to both DC and AC voltage signals:
  - For a *DC signal*, $\omega = 0\left(\frac{\omega}{\omega_b} = 0,\ \text{or}\ \frac{\omega_b}{\omega} = \infty\right)$, and thus $|V_o| = 0$ and $\phi = \frac{\pi}{2} = 90^0$. Thus $V_o = 0$.
    I.e. ***DC signals are completely cut off***.

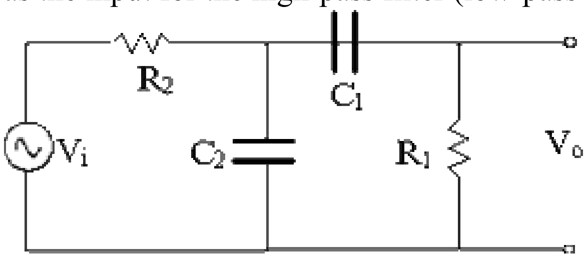
- For a *low frequency AC signal*, $\omega = \text{small}\left(\frac{\omega}{\omega_b} << 1\right)$, and thus $|V_o| \rightarrow 0$ and $\phi \approx \frac{\pi}{2} = 90^0$.
  Thus $V_o \approx 0$.
  I.e. ***low frequencies are attenuated or filtered by the circuit***, and have a ***90^o phase shift***.
- For a *high frequency AC signal*, $\omega = \text{large}\left(\frac{\omega}{\omega_b} >> 1\right)$, and thus $|V_o| \rightarrow |V_i|$ and $\phi \rightarrow 0$.
  Thus $|V_o| \rightarrow |V_i|$ with a small phase shift.
  I.e. ***high frequencies pass through the circuit, largely unaffected***. (This is why such a filter is called a high-pass filter.)
- For an *AC signal with a frequency exactly at the cutoff frequency*, $\omega = \omega_b \left(\frac{\omega}{\omega_b} = 1\right)$,
  and thus $|V_o| = \frac{|V_i|}{\sqrt{2}} \approx 0.707|V_i|$ and $\phi = \arctan(1) = \frac{\pi}{4} = 45°$. Thus V_o is attenuated by some 30% with a 45^o phase shift. I.e. ***the cutoff frequency is attenuated by a factor of*** $1/\sqrt{2}$.

- Summary plots for the high-pass filter are not shown here. They are basically opposite to those of the low-pass filter.
- Note, for example, that ***the output signal leads the input signal***, since the phase shift is positive rather than negative. In other words, for a high-pass filter, the peak of the output signal appears to occur *earlier* in time than that of the input signal, as sketched below.



First-order band-pass filter

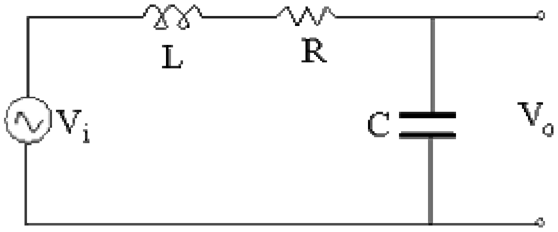
- A ***first-order band-pass filter*** circuit can be created by using the output of the low-pass filter as the input for the high-pass filter (low-pass filter and high-pass filter in *series*), as shown:



The left portion of the schematic diagram represents the low-pass filter, and the right portion represents the high-pass filter.
- Here there are two break frequencies to define, i.e. a low-pass break frequency and a high-pass break frequency,
  $\omega_{b1} = \frac{1}{R_1 C_1}$ and $\omega_{b2} = \frac{1}{R_2 C_2}$

Second-order low-pass filter

- An inductor can be added to the resistor and capacitor to create a higher-order low-pass filter circuit as shown:



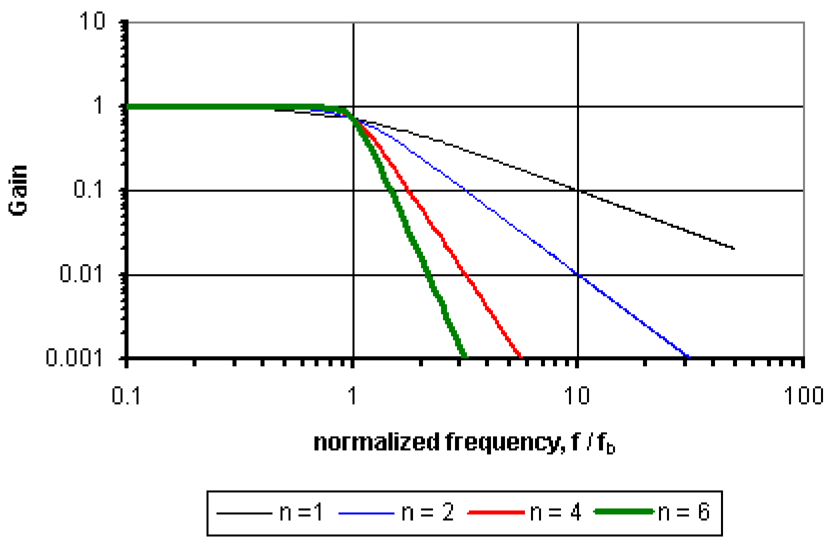
  In this case, it turns out that the filter behaves as a ***second-order*** low pass filter.
- See the text, which illustrates how higher-order filters can introduce some "wiggles" in the frequency response diagram.

Higher-order low-pass filters

- Butterworth filters of higher order can be constructed, usually with op amps rather than simply with resistors, capacitors, and inductors (see text for details).
- In general, for a Butterworth low-pass filter of order n, the gain is

$$G \equiv \frac{|V_o|}{|V_i|} = \frac{1}{\sqrt{1+\left(\frac{\omega}{\omega_b}\right)^{2n}}} = \frac{1}{\sqrt{1+\left(\frac{f}{f_b}\right)^{2n}}}$$

- Higher order filters have a much faster roll-off rate. For a given break frequency, this means that the filter attenuates high frequencies much better, as illustrated in the plot below for Butterworth filters of order 1, 2, 4, and 6:



- For example, using the same numbers as in the example above (f = 1000 Hz = noise and f_b = 50 Hz = break frequency), the gain of a 4th-order low-pass Butterworth filter is

$$G \equiv \frac{|V_o|}{|V_i|} = \frac{1}{\sqrt{1+\left(\frac{f}{f_b}\right)^{2n}}} = \frac{1}{\sqrt{1+\left(\frac{1000}{50}\right)^{2 \cdot 4}}} = 6.25 \times 10^{-6}$$

  Or, in terms of decibels, the gain is G_{dB} = -104. dB. Notice how much higher is this attenuation, compared to that of a first-order filter.

- Meanwhile, at the desired signal frequency of 10 Hz, the gain is 0.9999987, i.e. -0.000011 dB, which is negligible.
- Commercial Butterworth filters can be purchased with adjustable order, typically of order 1, 2, 4, 6, and 8.